

Gog and Magog triangles, and the Schützenberger involution

Hayat Cheballah and Philippe Biane

ABSTRACT. We describe an approach to finding a bijection between Alternating Sign Matrices and Totally Symmetric Self-Complementary Plane Partitions, which is based on the Schützenberger involution. In particular we give an explicit bijection between Gog and Magog trapezoids with two diagonals.

1. Introduction

1.1. Alternating Sign Matrices. An *alternating sign matrix* (ASM) is a square matrix with entries in $\{-1, 0, +1\}$ such that, on each line and on each column, the non zero entries alternate in sign, the sum of each line and each column being equal to 1. The number of such matrices of size n is

$$(1.1) \quad A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, \dots$$

Zeilberger [9], Kuperberg [6]. The full story is in [1].

It has been known for a long time that the numbers A_n also count the number of Totally Symmetric Self-Complementary Plane Partitions (TSSCPP), however no explicit bijection between these classes of objects has been constructed, and finding one is a major open problem in combinatorics.

In this paper we propose an approach to this question which is based on the Schützenberger involution. More precisely, we consider Gog and Magog triangles (in the terminology of D. Zeilberger), which are triangular arrays of positive integers, satisfying some growth conditions, in simple bijection with, respectively, ASMs and TSSCPPs. The basic idea underlying our approach is that these triangles are examples of Gelfand-Tsetlin patterns to which one can apply some known transformations, such as the Schützenberger involution. In fact we conjecture the existence of a bijection between Gog and Magog triangles which can be obtained in two steps, first by making a "modification" of a Gog triangle, based on its inversion pattern, then by applying the Schützenberger involution. This bijection should also preserve trapezoids, which are particular classes of triangles. As a first step towards a full bijection we construct here a bijection between $(n, 2)$ Gog and Magog trapezoids (the terminology is explained below).

The paper is organized as follows.

In section 2 we introduce the definitions of Gelfand-Tsetlin triangles, the Gog and Magog triangles and trapezoids, and the Schützenberger involution. Then, in section 3, we formulate a conjecture on the existence of a bijection between Gog

and Magog triangles preserving trapezoids. Finally in the last section we give a bijection between $(n, 2)$ Gog and Magog trapezoids.

2. Gog and Magog triangles and trapezoids

2.1. Gelfand-Tsetlin.

Definition 1. A Gelfand-Tsetlin triangle of size n is a triangular array $X = (x_{i,j})_{n \geq i \geq j \geq 1}$ of positive integers

$$\begin{array}{ccccccc} x_{n,1} & & x_{n,2} & & \dots & & x_{n,n-1} & & x_{n,n} \\ & x_{n-1,1} & & x_{n-1,2} & & \dots & & x_{n-1,n-1} & \\ & & \dots & & \dots & & \dots & & \\ & & & x_{2,1} & & x_{2,2} & & & \\ & & & & x_{1,1} & & & & \end{array}$$

such that, whenever the numbers below are defined,

$$x_{i+1,j} \leq x_{i,j} \leq x_{i+1,j+1}.$$

In other words, the triangle is made of n diagonals in the Northwest-Southeast (NW-SE) direction, of lengths $n, n-1, \dots, 2, 1$ (from left to right), and it is weakly increasing in the SE and in the NE directions.

Thus

$$\begin{array}{cccccc} & & 1 & & 2 & & 2 & & 3 & & 6 \\ & & & 1 & & 2 & & 2 & & 5 \\ & & & & 2 & & 2 & & 4 \\ & & & & & 2 & & 4 \\ & & & & & & 3 \end{array}$$

is a Gelfand-Tsetlin triangle of size 5.

Gog and Magog triangles will be obtained from Gelfand-Tsetlin triangles by imposing further conditions on the entries.

2.2. Gog.

2.2.1. Triangles.

Definition 2. A Gog triangle of size n is a Gelfand-Tsetlin triangle such that

$$(i) \quad x_{i,j} < x_{i,j+1}, \quad j < i \leq n-1$$

in other words, such that its rows are strictly increasing, and such that

$$(ii) \quad x_{n,j} = j, \quad 1 \leq j \leq n.$$

Here is an example with $n = 5$

$$\begin{array}{cccccc} & & 1 & & 2 & & 3 & & 4 & & 5 \\ & & & 1 & & 3 & & 4 & & 5 \\ & & & & 1 & & 4 & & 5 \\ & & & & & 2 & & 4 \\ & & & & & & 3 \end{array}$$

There is a simple bijection between Gog triangles and Alternating sign matrices (see e.g. [1]). If $(M_{ij})_{1 \leq i,j \leq n}$ is an ASM of size n , then the matrix $\tilde{M}_{ij} = \sum_{k=i}^n M_{kj}$ has exactly $i-1$ entries 0 and $n-i+1$ entries 1 on row i . Let $(x_{ij})_{j=1,\dots,i}$ be the columns (in increasing order) with a 1 entry of \tilde{M} on row $n-i+1$. The triangle $X = (x_{ij})_{n \geq i \geq j \geq 1}$ is the Gog triangle corresponding to M .

For example, the above Gog triangle corresponds to the following alternating sign matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

2.2.2. Trapezoids.

Definition 3. A (n, k) Gog trapezoid (for $k \leq n$) is a Gog triangle of size n , $X = (x_{i,j})_{n \geq i \geq j \geq 1}$ such that $x_{i,j} = j$ for $i - j \geq k$.

Below is a $(5, 2)$ Gog trapezoid.

$$\begin{array}{cccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 4 & & 5 \\ & & 1 & & 3 & & 4 \\ & & & 1 & & 3 \\ & & & & 2 \end{array}$$

See [5], [9].

2.3. Magog.

2.3.1. Triangles.

Definition 4. A Magog triangle of size n is a Gelfand-Tsetlin triangle such that

$$x_{i,i} \leq i, \quad 1 \leq i \leq n.$$

The set of Magog triangles of size n is in simple bijection with the set of Totally Symmetric Self Complementary Plane Partitions (see [1]).

2.3.2. Trapezoids.

Definition 5. A (n, k) Magog trapezoid (with $k \leq n$) is a Magog triangle $X = (x_{i,j})_{n \geq i \geq j \geq 1}$, such that $x_{i,j} = 1$ for $i - j \geq k$.

2.4. Schützenberger involution.

2.4.1. Gelfand-Tsetlin triangles label bases of irreducible representations of general linear groups. As such, they are in simple bijection with semi-standard Young tableaux. It follows that the Schützenberger involution, which is defined on SSYTs, can be transported to Gelfand-Tsetlin triangles. The following description of this involution has been studied by Berenstein and Kirillov [4].

First define operators s_k , for $k \leq n - 1$, acting on the set of Gelfand-Tsetlin triangles of size n . If $X = (x_{i,j})_{n \geq i \geq j \geq 1}$ is such a triangle the action of s_k on X is given by $s_k X = (\tilde{x}_{i,j})_{n \geq i \geq j \geq 1}$ with

$$\begin{aligned} \tilde{x}_{i,j} &= x_{i,j}, & \text{if } i \neq k \\ \tilde{x}_{k,j} &= \max(x_{k+1,j}, x_{k-1,j-1}) + \min(x_{k+1,j+1}, x_{k-1,j}) - x_{i,j} \end{aligned}$$

It is understood that $\max(a, b) = \max(b, a) = a$ and $\min(a, b) = \min(b, a) = a$ if the entry b of the triangle is not defined. The geometric meaning of the transformation of an entry is the following: on row k , any entry $x_{k,j}$ is surrounded by four (or less if it is on the boundary) numbers, increasing from left to right.

$$\begin{array}{ccc} & x_{k+1,j} & x_{k+1,j+1} \\ & x_{k,j} & \\ x_{k-1,j-1} & & x_{k-1,j} \end{array}$$

These four numbers determine a smallest interval containing $x_{k,j}$, namely

$$[\max(x_{k+1,j}, x_{k-1,j-1}), \min(x_{k+1,j+1}, x_{k-1,j})]$$

and the transformation maps $x_{k,j}$ to its symmetric with respect to the center of this interval.

Define $\omega_j = s_j s_{j-1} \dots s_2 s_1$.

Definition 6. *The Schützenberger involution, acting on Gelfand-Tsetlin triangles of size n , is given by the formula*

$$S = \omega_1 \omega_2 \dots \omega_{n-1}$$

It is a non trivial result that S is an involution [4]; beware that the s_k do not satisfy the braid relations.

2.4.2. One can compute the rightmost diagonal of SX .

Lemma 1. *Let $X = (X_{i,j})$ be a Gelfand-Tsetlin triangle and $Y = SX$ its image by the Schützenberger involution, then*

$$(2.1) \quad Y_{nn} = X_{nn}$$

$$(2.2) \quad Y_{kk} =$$

$$\max_{n=j_0 > j_1 > j_2 \dots > j_{n-k} \geq 1} \left[\left(\sum_{i=0}^{n-k-1} X_{j_i+i, j_i} - X_{j_{i+1}+i, j_{i+1}} \right) + X_{j_{n-k}+n-k, j_{n-k}} \right]$$

for $k < n$

Proof We recall the description of the Schützenberger involution in terms of words and the Robinson-Schensted correspondance. To the Gelfand-Tsetlin triangle X let us associate the semi-standard Young tableau, with entries in $[1, n]$, such that the shape of the tableau formed with letters $u \leq i$ is the partition X_{ij} , $j = 1 \dots i$. For example, our Gelfand-Tsetlin triangle

$$\begin{array}{cccccc} 1 & 2 & 2 & 3 & 6 \\ & 1 & 2 & 2 & 5 \\ & & 2 & 2 & 4 \\ & & & 2 & 4 \\ & & & & 3 \end{array}$$

corresponds to the tableau (in French notation)

$$\begin{array}{cccccc} 5 \\ 4 & 5 \\ 3 & 3 \\ 2 & 2 & 5 \\ 1 & 1 & 1 & 2 & 4 & 5 \end{array}$$

To such a tableau we associate the word w obtained by reading the tableau from top to bottom and from left to right, thus

$$w = 54533225111245$$

in our example. Then we perform the Schützenberger involution on the word: we read it backwards and replace each letter i by $n+1-i$ to give a word Sw , in our example

$$Sw = 12455514433121.$$

Observe that this word is a concatenation of nondecreasing words $(Sw)_1, (Sw)_2, \dots$ corresponding to the successive rows, and that these nondecreasing words, viewed as partitions, are the partitions conjugate to the successive SW-NE diagonals of the original Gelfand-Tsetlin triangle.

Applying the Robinson-Schensted algorithm on the word Sw yields an insertion tableau which is the image of our tableau by the Schützenberger involution. It is well known that the longest part of the tableau thus obtained is equal to the longest nondecreasing subsequence of the word. Thus the largest element of the top row is unchanged. Moreover, the largest element of the i^{th} row (from bottom) is equal to the length of the longest nondecreasing subsequence of the subword of Sw made of numbers $\leq i$. Now $X_{nn} - X_{j_1 j_1}$ is the length of the part of the first nondecreasing subword $(Sw)_1$ of Sw made of letters $\leq n - j_1$, then $X_{j_1+1, j_1} - X_{j_2+1, j_2}$ is the length of the part of the second nondecreasing subword $(Sw)_2$ of Sw made of letters in $[n - j_1, n - j_2]$, and so on. This yields formula (2.2). \square

2.4.3. GOGAm triangles. Since the Schützenberger involution consists in reading a word backwards and inverting the letters, we introduce the following definition.

Definition 7. A GOGAm triangle of size n is a Gelfand-Tsetlin triangle such that its image by the Schützenberger involution is a Magog triangle of size n .

Thanks to Lemma 1 we can give a description of GOGAm triangles.

Proposition 1. $X = (X_{i,j})$ be a Gelfand-Tsetlin triangle then X is a GOGAm triangle if and only if $X_{nn} \leq n$ and, for all $1 \leq k \leq n - 1$, and all $n = j_0 > j_1 > j_2 \dots > j_{n-k} \geq 1$, one has

$$\left(\sum_{i=0}^{n-k-1} X_{j_i+i, j_i} - X_{j_{i+1}+i, j_{i+1}} \right) + X_{j_{n-k}+n-k, j_{n-k}} \leq k$$

Proof. Immediate from Lemma 1.

2.4.4. GOGAm trapezoids.

Definition 8. A (n, k) GOGAm trapezoid is a GOGAm triangle of size n such that $x_{i,j} = 1$ for $i - j \geq k$. Equivalently, it is the image by the Schützenberger involution of a (n, k) Magog trapezoid.

2.5. A conjecture. We are now in position to state our conjecture on the Gog-Magog bijection.

Conjecture 1. There exists a bijection from Gog triangles of size n to Magog triangles of size n , which maps (n, k) Gog trapezoids to (n, k) Magog trapezoids for all $k \leq n$.

A similar conjecture has been made by Krattenthaler [5]. One can refine the conjecture by considering statistics on Gog and Magog triangles. Such a study will be made in [2], where some further motivation for considering the Schützenberger involution will be provided. We consider one such statistic in section 3.5.3 below.

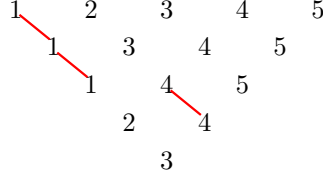
In order to construct such a bijection, it is enough to construct a bijection between Gog triangles and GOGAm triangles of the same size. In the next section we will give a bijection between $(n, 2)$ Gog trapezoids and $(n, 2)$ GOGAm trapezoids, which restricts to a bijection between $(n, 1)$ Gog trapezoids and $(n, 1)$ GOGAm trapezoids.

3. $(n, 2)$ Gog and Magog trapezoids

3.1. Inversions.

Definition 9. An inversion in a Gog triangle is a pair (i, j) such that $x_{i,j} = x_{i+1,j}$.

For example the following Gog triangle contains three inversions, $(2, 2)$, $(3, 1)$, $(4, 1)$, the respective equalities being in red on this picture.

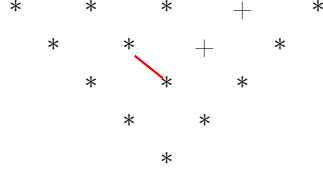


Remark 1. The number of inversions of a Gog triangle coincides with the number of inversions of its associated ASM, as defined by Mills, Robbins, Rumsey [8].

Definition 10. Let $X = (x_{i,j})_{n \geq i \geq j \geq 1}$ be a Gog triangle and let (i, j) be such that $1 \leq i \leq j \leq n$.

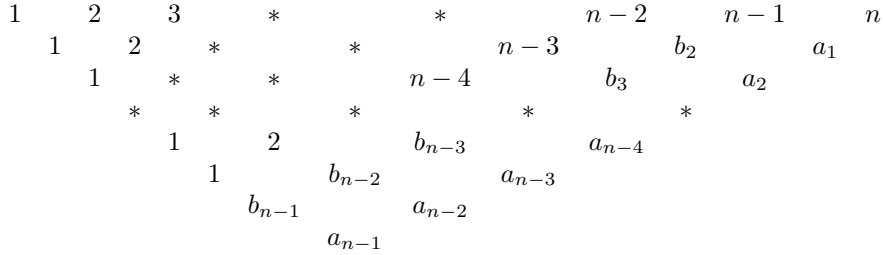
An inversion (k, l) covers (i, j) if $i = k + p$ and $j = l + p$ for $1 \leq p \leq n - k$.

The entries (i, j) covered by an inversion are depicted with "+" on the following picture.



The basic idea for our bijection is that for any inversion in the Gog triangle we should subtract 1 to the entries covered by this inversion. This simple minded procedure works for $(n, 1)$ trapezoids, as we will show as a byproduct of our bijection for $(n, 2)$ trapezoids. It is a good exercise to check this directly. The procedure does not work for (n, k) trapezoids with $k > 1$ but, by making some adequate adaptations, we will obtain a bijection for trapezoids of size $(n, 2)$.

3.2. $(n, 2)$ trapezoids. Consider a $(n, 2)$ Gog trapezoid. This is an array of the form



We shall give an algorithm which builds a GOGAm triangle from the Gog triangle by successively adding NW-SE diagonals of increasing lengths, and making

appropriate changes to the triangle. In the end we will obtain a triangle of the form

$$\begin{array}{cccccccccccc}
 1 & & 1 & & 1 & & * & & * & & 1 & & \beta_1 & & \alpha_0 \\
 & 1 & & 1 & & * & & * & & 1 & & \beta_2 & & \alpha_1 \\
 & & 1 & & * & & * & & * & & \beta_3 & & \alpha_2 \\
 & & & * & & * & & * & & * & & * & & \\
 & & & & 1 & & 1 & & \beta_{n-3} & & \alpha_{n-4} & & & \\
 & & & & & 1 & & \beta_{n-2} & & \alpha_{n-3} & & & & \\
 & & & & & & \beta_{n-1} & & \alpha_{n-2} & & & & & \\
 & & & & & & & \alpha_{n-1} & & & & & &
 \end{array}$$

By Proposition 1, such a triangle is a GOGAm triangle if and only if

$$\begin{aligned}
 \alpha_0 &\leq n \\
 \alpha_0 - \alpha_i + \beta_i &\leq n - 1 \quad \text{for } 1 \leq i \leq n - 1 \\
 \alpha_0 - \alpha_i + \beta_i - \beta_j + 1 &\leq j - 1 \quad \text{for } 1 \leq i < j \leq n - 1
 \end{aligned}$$

3.3. The algorithm. First step: the rightmost NW-SE diagonal consists of one entry n and is not changed, yielding the triangle of size 1 equal to $X^{(1)} = n$.

Second step: The triangle formed by the two first diagonals is

$$\begin{array}{cc}
 n - 1 & n \\
 & a_1
 \end{array}$$

where $a_1 = n$ or $n - 1$. In the first case, the algorithm yields the triangle

$$X^{(2)} = \begin{array}{cc} n - 1 & n \\ & n \end{array}$$

in the second case we have an inversion and accordingly subtract 1 from the upper right entry, which gives the triangle

$$X^{(2)} = \begin{array}{cc} n - 1 & n - 1 \\ & n - 1 \end{array}$$

Assume now that the first k diagonals have been treated and a triangle $X^{(k)}$ of size k , of the form

$$\begin{array}{cccccccccccc}
 n - k + 1 & & & & n - k + 1 & & * & & & & n - k + 1 & & v_1 & & u_0 \\
 & n - k + 1 & & & & & * & & & & * & & v_2 & & u_1 \\
 & & & * & & & * & & & & * & & * & & \\
 & & & & n - k + 1 & & & v_{k-2} & & & & u_{k-3} & & & \\
 & & & & & v_{k-1} & & & u_{k-2} & & & & & & \\
 & & & & & & u_{k-1} & & & & & & & &
 \end{array}$$

has been obtained. Furthermore assume that this triangle satisfies the inequalities

$$(3.1) \quad u_0 \leq n$$

$$(3.2) \quad u_0 - u_i + v_i \leq n - 1 \quad \text{for } 1 \leq i \leq k - 1$$

$$(3.3) \quad u_0 - u_i + v_i - v_j + 1 \leq j - 1 \quad \text{for } 1 \leq i < j \leq k - 1$$

and that

$$(3.4) \quad u_{k-1} = a_{k-1}.$$

Let us add, on the left of this triangle, the diagonal

$$\begin{array}{ccccccc}
 & & n-k & & & & \\
 & & & n-k & & & \\
 & & & & * & & \\
 & & & & & n-k & \\
 & & & & & & v_k \\
 & & & & & & & u_k
 \end{array}$$

with $u_k = a_k$, $v_k = b_k$. This yields a triangle $Z^{(k)}$ of size $k+1$ (this triangle will not, in general, be a Gelfand-Tsetlin triangle, because the inequality $v_k \leq v_{k-1}$ may be broken). The algorithm will modify $Z^{(k)}$ to get a triangle $X^{(k+1)}$ of size $k+1$, of the form

$$\begin{array}{cccccccccccc}
 n-k & & n-k & & * & & * & & n-k & & v'_1 & & u'_0 \\
 & n-k & & * & & * & & * & & v'_2 & & u'_1 \\
 & & * & & * & & * & & * & & u'_2 \\
 & & & * & & * & & * & & * \\
 & & & & n-k & & v'_{k-1} & & u'_{k-2} \\
 & & & & & v'_k & & u'_{k-1} \\
 & & & & & & u'_k
 \end{array}$$

We will check that the new triangle is a Gelfand-Tsetlin triangle and that (3.1), ..., (3.4) are verified for this new triangle. The modification will depend on the inversion pattern in the leftmost diagonal that we have added. In all cases, we will have

$$(3.5) \quad u'_k = u_k$$

the remaining entries being modified as follows, according to the four possibilities for the inversions in the two bottom rows.

(i) The first case is $v_k = n-k$, $u_k = n-k$, when there are two inversions. Then the modification consists in subtracting 1 from each of the entries of the previous triangle, that is we put $u'_i = u_i - 1$, $v'_i = v_i - 1$, for $i \leq k-1$, and $v'_k = v_k = n-k$.

(ii) The second is the case $v_k = n-k < u_k$. Then we put $u'_i = u_i$, $v'_i = v_i - 1$, for $i \leq k-1$, and $v'_k = v_k = n-k$.

(iii) The third case is when $n-k < v_k = u_k$. We put $u'_i = u_i - 1$ for $i \leq k-1$. Observe that $v_k = b_k = u_k < a_{k-1} = u_{k-1}$, therefore u_i , $0 \leq i \leq k$ is nonincreasing. Two cases occur:

(iiia) if the triangle we obtain is a Gelfand-Tsetlin triangle, then we keep it as the modified triangle, i.e. we put $v'_i = v_i$ for $i \leq k$.

(iiib) if the triangle is not Gelfand-Tsetlin, then there must exist $j \leq k-1$ with $v_j = u_j$. In this case, we put $v'_i = v_i - 1$, for $i \leq k-1$, and we put $v'_k = n-k$.

(iv) Finally the last case is when $n-k < v_k < u_k$. There are two possibilities.

(iva) if $v_k \leq v_{k-1}$, then $Z^{(k)}$ is a Gelfand-Tsetlin triangle, and we do not modify it, i.e. we put $u'_i = u_i$, $v'_i = v_i$ for all $i \leq k$, thus $X^{(k+1)} = Z^{(k)}$.

(ivb) The last case is $v_k > v_{k-1}$. First we put $u'_i = u_i$ for all i . Let

$$(3.6) \quad l = \max\{i | v_{k-i} \leq v_k - i\}$$

Since v_{k-i} is nondecreasing and $v_k - i$ is decreasing, one has $l \geq 1$ and $v_{k-i} \leq v_k - i$ for all $i \leq l$. We put $v'_k = v'_{k-1} = \dots = v'_{k-l+1} = n - k$ and $v'_{k-l} = v_k - l$, all the other entries being unchanged: $v'_i = v_i$ for $i < k - l$.

Remark 2. Rules (i), (ii), (iiia), (iva) consist just in subtracting 1 from entries covered by the inversions in the SE-NW diagonal which has been added. The rules (iiib) and (ivb) are more subtle.

3.3.1. *Proof of the algorithm, first part.* Let us now check that, in each case, we obtain a Gelfand-Tsetlin triangle $X^{(k)}$ satisfying inequalities (3.1), (3.2), (3.3) (the identity (3.4) is immediate from (3.5)).

We start with rules (i), (ii), (iiia), (iiib), (iva).

(i) Since $a_{k-1} = u_{k-1} > v_k$ and $v_{k-1} \geq n - k + 1$, $X^{(k+1)}$ is a Gelfand-Tsetlin triangle. For $1 \leq i < j \leq k - 1$ one has $u'_0 - u'_i + v'_i - v'_j = u_0 - u_i + v_i - v_j$ hence (3.3) is satisfied for these values. Since

$$u'_0 - u'_i + v'_i - v'_k = u_0 - u_i + v_i - 1 - (n - k) \leq n - 1 - 1 - (n - k) = k - 2$$

we see that (3.3) is satisfied for all values. Since $u'_0 = u_0 - 1 \leq n - 1$ and $u'_i \geq v'_i$ one has (3.2) and (3.1).

(ii) Again, $X^{(k+1)}$ is clearly a Gelfand-Tsetlin triangle. For $1 \leq i < j \leq k$ we check (3.3) as above, while (3.1) is clear, finally $u'_0 - u'_i + v'_i = u_0 - u_i + v_i - 1 \leq n - 2$, and $u'_0 - u'_k + v'_k \leq n - 1$ since $-u'_k + v'_k \leq -1$, which gives (3.2).

(iiia) Since $u_i > v_i$, one has $u'_i \geq v'_i$ for $i \leq k$, and the triangle $X^{(k+1)}$ is Gelfand-Tsetlin triangle.

One has, successively,

$$\begin{aligned} u'_0 &= u_0 - 1 \\ u'_0 - u'_i + v'_i &= u_0 - u_i + v_i \quad i < k \\ u'_0 - u'_k + v'_k &= u_0 - 1 \leq n - 1 \\ u'_0 - u'_i + v'_i - v'_j &= u_0 - u_i + v_i - v_j \quad i < j < k \\ u'_0 - u'_i + v'_i - v'_k &< n - 1 - (n - k) = k - 1 \quad (\text{since } v'_k > n - k) \end{aligned}$$

from which inequalities (3.1), (3.2), (3.3) follow.

(iiib) The new triangle is clearly Gelfand-Tsetlin. Furthermore, one has

$$\begin{aligned} u'_0 &= u_0 - 1 \\ u'_0 - u'_i + v'_i &= u_0 - u_i + v_i - 1 \quad i < k \\ u'_0 - u'_k + v'_k &< u'_0 \leq n \\ u'_0 - u'_i + v'_i - v'_j &= u_0 - u_i + v_i - v_j \quad i < j < k \\ u'_0 - u'_i + v'_i - v'_k &= u_0 - u_i + v_i - 1 - (n - k) \leq k - 2 \end{aligned}$$

which imply inequalities (3.1), (3.2), (3.3).

(iva) The fact that $X^{(k+1)}$ is Gelfand-Tsetlin is immediate. The inequalities are preserved, indeed, all inequalities involving indices $< k$ are immediate, and one

has

$$\begin{aligned} u'_0 - u'_k + v'_k &\leq u'_0 - 1 \leq n - 1 \quad \text{since } u'_k > v'_k \\ u'_0 - u'_i + v'_i - v'_k &\leq n - 1 - (n - k + 1) = k - 2 \quad \text{since } v'_k > n - k \end{aligned}$$

3.3.2. Proof of the algorithm, second part. We now consider the last rule, (ivb). This is the most delicate part of the proof. We first gather some information on the algorithm which has been constructed up to now.

Lemma 2. *Just after a step where rule (i) or (ii) is applied, rule (iiib) never applies.*

Proof. Suppose that rule (i) applies to $Z^{(k)}$, then $n - k = b_k = v_k = a_k = u_k$, and $n - k - 1 < b_{k+1} = a_{k+1}$ is impossible since this would yield $b_{k+1} \geq a_k$ contradicting the Gog strict inequality for the original triangle. If rule (ii) applies to $Z^{(k)}$ then $v_i < u_i$ in $X^{(k+1)}$, for all $i < k$, therefore rule (iiib) cannot be applied to $Z^{(k+1)}$. \square

Lemma 3. *If rule (ivb) applies at step k , then necessarily at the previous step either rule (iiib) or (ivb) was applied.*

Proof. If one of the other rules had been applied at the previous step, one would have $v_{k-1} \geq v_k$. \square

Lemma 4. *If rule (ivb) is applied to the triangle $Z^{(k)}$, then to each of the triangles $Z^{(k-l)}, Z^{(k-l+1)}, \dots, Z^{(k-1)}$ either rule (iiib) or (ivb) was applied.*

Proof. Assume that at some step $t < k$ in the algorithm we have applied rule (iiia) or (iva) to $Z^{(t)}$, then the entry $v_t^{(t+1)}$ in the triangle $X^{(t+1)}$ (we emphasize the dependence on the step by adding a superscript) satisfies $b_t = v_t^{(t+1)}$. At each next step s , we will subtract at most 1 from $v_t^{(s)}$, therefore, in the triangle $Z^{(k)}$,

$$v_t^{(k)} \geq b_t - (k - t - 1) \geq b_k - (k - t - 1) = v_k^{(k)} - (k - t - 1) > v_k^{(k)} + t - k$$

It follows that, in $Z^{(k)}$, one has $l < k - t$ (where l is defined by (3.6)). We conclude that, to each of the triangles $Z^{(k-l)}, Z^{(k-l+1)}, \dots, Z^{(k-1)}$ either rule (i), (ii), (iiib) or (ivb) was applied. But we have seen that rule (iiib) cannot follow immediately rule (i) or (ii) and that rule (ivb) always follows either rule (iiib) or (ivb), so that in fact only rule (iiib) or (ivb) has been applied to each of the triangles $Z^{(k-l)}, Z^{(k-l+1)}, \dots, Z^{(k-1)}$. \square

Lemma 5. *If rule (ivb) is applied to the triangle $Z^{(k)}$, then one has*

$$v_{k-1} = \dots = v_{k-l} = n - k + 1.$$

Proof. Since $n - k + 1 \leq v_{k-1} \leq \dots \leq v_{k-l}$ it is enough to prove that $v_{k-l} \leq n - k + 1$. By the preceding Lemma, either rule (iiib) or (ivb) has been applied to the triangles $Z^{(k-l)}, Z^{(k-l+1)}, \dots, Z^{(k-1)}$. Let us look at the successive values of the entry $v_{k-l}^{(s)}$ in the triangle $X^{(s)}$ (or $Z^{(s)}$). One has $v_{k-l}^{(k-l+1)} = n - k + l$, since rule (iiib) or (ivb) has been applied to $Z^{(k-l)}$. Each time rule (iiib) is applied $v_{k-l}^{(s)}$ is decreased by 1. There are two cases

(a) If only rule (iiib) is applied to $Z^{(k-l)}, Z^{(k-l+1)}, \dots, Z^{(k-1)}$ then one has $v_{k-l}^{(k)} = n - k + 1$.

(b) If not, let i be the least index $l \geq i \geq 1$ such that rule (ivb) is applied to $Z^{(k-i)}$, and let $l' = \max\{j | v_{k-i}^{(k-i)} - j \geq v_{k-i-j}^{(k-i)}\}$. By rule (ivb) one has

$$v_{k-l'-i}^{(k-i+1)} = b_{k-i} - l', \quad v_{k-i-j}^{(k-i+1)} = n - k + i, \quad j = 0, 1, \dots, l' - 1$$

Since rule $(iiib)$ is applied to $Z^{(k-i+1)}, \dots, Z^{(k-1)}$, one has $v_{k-l'-i}^{(k)} = b_{k-i} - l' - i + 1$ and

$$(3.7) \quad v_{k-p}^{(k)} = n - k + 1, \quad p = 1, 2, \dots, l' + i - 1$$

it follows that

$$v_{k-l'-i}^{(k)} = b_{k-i} - l' - i + 1 \geq b_k - l' - i + 1 = v_k^{(k)} - l' - i + 1$$

hence, by (3.6),

$$v_{k-l'-i}^{(k)} > v_k^{(k)} - l' - i$$

therefore $l < l' + i$, and $v_{k-l} = n - k + 1$ by (3.7). \square

Lemma 6. *If rule $(iiib)$ or (ivb) is applied to the triangle $Z^{(k)}$, then there exists some $i < k - l$ such that $u'_i = v'_i$.*

Proof. For rule $(iiib)$ this is easy to see.

In the case of rule (ivb) , there exists some step before k , when rule $(iiib)$ has been applied and then only rules $(iiib)$ or (ivb) have been applied. If rule $(iiib)$ is applied, there must exist an i with $u_i = v_i$, and then applying either rule $(iiib)$ or (ivb) cannot destroy this pair $u_i = v_i$. This implies that there exists some i such that $u'_i = v'_i$. Such a pair cannot exist for $i \geq k - l$ by the preceding lemma, therefore $i < k - l$. \square

3.3.3. Proof of the algorithm, end. Assuming that rule (ivb) is applied to the triangle $Z^{(k)}$, we can now check that our triangle $X^{(k+1)}$ satisfies all the required properties. Since $v'_{k-l} = v_k - l$, and $v_{k-l-1} > v_k - l - 1$, by the definition of l , one has $v'_{k-l-1} \geq v'_{k-l}$. This implies that $X^{(k+1)}$ is a Gelfand-Tsetlin triangle, as is easily verified.

Let us check the inequalities (3.1), (3.2), (3.3).

First, since $u'_0 = u_0$, (3.1) is clear. Consider $u'_0 - u'_i + v'_i$. Since $u'_i = u_i$ is unchanged and $v'_i \leq v_i$ for all values of i , except v'_{k-l} , in order to check (3.2) it is enough to consider $u'_0 - u'_{k-l} + v'_{k-l}$ and $u'_0 - u'_k + v'_k$. One has

$$u'_{k-l} = u_{k-l} \geq u_k > v_k - l = v'_{k-l}$$

therefore

$$u'_0 - u'_{k-l} + v'_{k-l} \leq n - 1.$$

Since $u'_k > v'_k$, one has

$$u'_0 - u'_k + v'_k \leq n - 1.$$

Consider $u'_0 - u'_i + v'_i - v'_j$, for $i < j \leq k$.

If $j < k - l$, then $u'_0 - u'_i + v'_i - v'_j = u_0 - u_i + v_i - v_j$, so (3.3) is preserved.

If $j = k - l$, then $u'_i = u_i$, $v'_i = v_i$, $v'_j \geq v_j$ therefore the inequality is again true.

If $j > k - l > i$, then $v'_j = n - k = v_{k-l} - 1$ (by Lemma 5), therefore

$$u'_0 - u'_i + v'_i - v'_j = u_0 - u_i + v_i - v_{k-l} + 1 \leq k - l - 1 \leq j - 2.$$

If $j > k - l = i$ then

$$\begin{aligned} u'_0 - u'_i + v'_i - v'_j &= u_0 - u_{k-l} + v_k - l - n + k \\ &= u_0 - n + v_k - u_{k-l} - l + k \leq k - l - 1 \leq j - 2 \end{aligned}$$

since $v_k < u_{k-l}$.

If $k > j > i > k - l$ then $v'_i - v'_j = v_i - v_j$ and $u'_0 - u'_i = u_0 - u_i$ therefore

$$u'_0 - u'_i + v'_i - v'_j = u_0 - u_i + v_i - v_j \leq j - 2.$$

Finally if $k = j > i > k - l$, then

$$u'_0 - u'_i + v'_i - v'_k = u_0 - u_i + v_i - 1 - (n - k) \leq n - 1 - 1 - (n - k) = k - 2. \quad \square$$

Applying the algorithm until we have treated all diagonals, we obtain thus a $(n, 2)$ GOGAm trapezoid from our $(n, 2)$ Gog trapezoid.

3.3.4. Invertibility. We can infer from the leftmost SE-NW diagonal of $X^{(k+1)}$ which rule was applied to $Z^{(k)}$. The only ambiguity is whether rule (ii), (iiib) or (ivb) has been applied when $n - k = v'_k < u'_k$. Rule (ii) has been applied if and only if one has $u'_i > v'_i$ for all $i < k$. In order to distinguish between rules (iiib) and (ivb) we now state the following lemma.

Lemma 7. Assume $X^{(k+1)}$ is obtained from $Z^{(k)}$ by applying rule (iiib) or (ivb), and let $l = 1 + \max\{i | v'_{k-i} = n - k\}$, then

- (a) $v'_{k-l} + l < u'_k$ if rule (ivb) has been applied.
- (b) $v'_{k-l} + l \geq u'_k$ if rule (iiib) has been applied.

Proof. Part (a) is obvious from the statement of rule (ivb), since $v'_{k-l} + l = v_k < u_k = u'_k$.

In order to prove part (b), note that in case (iiib) is applied to $Z^{(k)}$, then by Lemma 2, to all the triangles $Z^{(k-i)}$ for $1 \leq i \leq l - 1$ either rule (iiib) or (ivb) has been applied. If only rule (iiib) has been applied to $Z^{(k-l+1)}, \dots, Z^{(k-1)}$, then rule (iiia) or (iva) must have been applied to $Z^{(k-l)}$, therefore, $v'_{k-l} = b_{k-l} - l$ which implies $v'_{k-l} + l = b_{k-l} \geq b_k = a_k = u'_k$.

If rule (ivb) has been applied at some step t with $k - l + 1 \leq t \leq k - 1$, then let i be the smallest number such that (ivb) has been applied to $Z^{(k-i)}$. By Lemma 5 there exists an $l' \geq 1$ such that

$$v_{k-i}^{(k-i+1)} = \dots v_{k-i-l'+1}^{(k-i+1)} = n - k + i - 1$$

and

$$v_{k-i-l'}^{(k-i+1)} = b_{k-i} - l' > n - k + i - 1.$$

Since rule (iiib) is applied to $Z^{(k-i+1)}, \dots, Z^{(k-1)}$ it follows that

$$v'_k = \dots = v'_{k-i-l'-1} = n - k$$

and

$$v'_{k-i-l'} = b_{k-i} - l' - i + 1 > n - k$$

therefore $l = l' + i$ and $v'_{k-l} + l \geq u'_k$ since $v'_{k-l} + l = b_{k-i} \geq b_k = a_k = u'_k$. \square

3.4. The inverse map.

3.4.1. *The algorithm.* We now prove that the map defined above has an inverse. Let X be a $(n, 2)$ GOGAm trapezoid of shape

$$\begin{array}{cccccccccccc}
 1 & & 1 & & 1 & & * & & * & & 1 & & \beta_1 & & \alpha_0 \\
 & 1 & & 1 & & * & & * & & 1 & & \beta_2 & & \alpha_1 \\
 & & 1 & & * & & * & & * & & \beta_3 & & \alpha_2 \\
 & & & * & & * & & * & & * & & * & & \\
 & & & & 1 & & 1 & & \beta_{n-3} & & \alpha_{n-4} & & & \\
 & & & & & 1 & & \beta_{n-2} & & \alpha_{n-3} & & & & \\
 & & & & & & \beta_{n-1} & & \alpha_{n-2} & & & & & \\
 & & & & & & & \alpha_{n-1} & & & & & &
 \end{array}$$

One has

$$\begin{aligned}
 \alpha_0 &\leq n \\
 \alpha_0 - \alpha_i + \beta_i &\leq n - 1 \quad \text{for } 1 \leq i \leq n - 1 \\
 \alpha_0 - \alpha_i + \beta_i - \beta_j + 1 &\leq j - 1 \quad \text{for } 1 \leq i < j \leq n - 1
 \end{aligned}$$

We shall give an algorithm which is the inverse of the one above.

Let k be an integer decreasing from $k = n - 1$ to $k = 0$. Let $Y^{(n)}$ be an empty set, and $X^{(n)} = X$; at each step we will have a pair $(Y^{(k+1)}, X^{(k+1)})$ where $Y^{(k+1)}$ is an array (non empty only for $k < n - 1$)

$$\begin{array}{cccccccc}
 1 & & 2 & & * & & n - k - 1 \\
 & 1 & & 2 & & * & \\
 & & * & & * & & * & n - k - 1 \\
 & & & * & & * & & b_{k+1} \\
 & & & & * & & * & a_{k+1} \\
 & & & & & 1 & & b_{n-2} \\
 & & & & & & b_{n-1} & a_{n-2} \\
 & & & & & & & a_{n-1}
 \end{array}$$

which forms the leftmost NW-SE diagonals of a Gog triangle, and $X^{(k+1)}$ is a Gelfand-Tsetlin triangle:

$$\begin{array}{cccccccc}
 n - k & & n - k & & * & & n - k & & v'_1 & & u'_0 \\
 & n - k & & n - k & & * & & v'_2 & & u'_1 \\
 & & * & & * & & * & & * & & \\
 & & & n - k & & v'_{k-1} & & u'_{k-2} & & & \\
 & & & & v'_k & & u'_{k-1} & & & & \\
 & & & & & u'_k & & & & &
 \end{array}$$

satisfying the inequalities (3.1), (3.2), (3.3). Then we make a modification of the triangle $X^{(k+1)}$, according to the rules below, to get a triangle $Z^{(k)}$

$$\begin{array}{cccccccc}
 n - k & & n - k + 1 & & * & & n - k + 1 & & v_1 & & u_0 \\
 & n - k & & n - k + 1 & & * & & v_2 & & u_1 \\
 & & * & & * & & * & & * & & \\
 & & & n - k & & v_{k-1} & & u_{k-2} & & & \\
 & & & & v_k & & u_{k-1} & & & & \\
 & & & & & u_k & & & & &
 \end{array}$$

Then we add the leftmost NW-SE diagonal of this triangle to the right of $Y^{(k+1)}$ to get $Y^{(k)}$ (thus $b_k = v_k$ and $a_k = u_k$), and take the remaining triangle as $X^{(k)}$. We will prove that, at each step, $X^{(k)}$ is a Gelfand-Tsetlin triangle which satisfies the inequalities (3.1), (3.2), (3.3). Also we will prove that, at the next step of the algorithm the entries a_{k-1}, b_{k-1} satisfy

$$(3.8) \quad n - k + 1 \leq b_{k-1}, \quad b_k \leq b_{k-1}, \quad b_k < a_{k-1}, \quad b_k \leq a_k \leq a_{k-1} \leq n$$

which imply that the triangle $Y^{(0)}$ is a Gog triangle.

We will use the following notation: if $v'_k = n - k$ and there exists $i < k$ such that $u'_i = v'_i$, then

$$(3.9) \quad l = 1 + \max\{j | v'_{k-j} = n - k\}.$$

Let us now describe the modification map yielding triangle $Z^{(k)}$ from $X^{(k+1)}$ by the inverse algorithm, for which we consider several cases, inverse to the cases considered in the direct algorithm.

(i) $n - k = v'_k = u'_k$, then we put $u_i = u'_i + 1, v_i = v'_i + 1$ for $i \leq k - 1$ and $v_k = v'_k, u_k = u'_k$.

(ii) The second case is $n - k = v'_k < u'_k$, and $v'_i < u'_i$ for all $i < k$. Then we put $u_i = u'_i, v_i = v'_i + 1$, for $i \leq k - 1$, and $v_k = v'_k, u_k = u'_k$.

(iii) $n - k < v'_k = u'_k$, then we put $u_i = u'_i + 1, v_i = v'_i$ for $i \leq k - 1$, and $v_k = v'_k, u_k = u'_k$.

(iiib) $n - k = v'_k < u'_k$, there exists $i < k$ such that $u'_i = v'_i$, and $v'_{k-l} + l \geq u'_k$ (recall (3.9)), then we put $u_i = u'_i + 1, v_i = v'_i + 1$, for $i \leq k - 1$, and $v_k = u_k = u'_k$.

(iva) $n - k < v'_k < u'_k$, then we put $u_i = u'_i, v_i = v'_i, i \leq k$.

(ivb) $n - k = v'_k < u'_k$, there exists $i < k$ such that $u'_i = v'_i$, and $v'_{k-l} + l < u'_k$, then we put $u_i = u'_i$, for $i \leq k, v_i = n - k + 1$ for $k - l \leq i \leq k - 1$, $v_k = v'_{k-l} + l$, and $v_i = v'_i$ for all other i .

Let us now check that this map is well defined. By section (3.3.4), it is an inverse of our modification map. We consider successively the cases (i), ..., (iv) above. First, by checking all cases successively, one sees that the sequence a_i constructed by the rules above is nonincreasing ($a_i \leq a_{i-1}$), and that $b_i \geq n - i$. The remaining inequalities in (3.8) will be checked case by case. We also have to check that the triangles $X^{(k)}$ are Gelfand-Tsetlin, and that they satisfy (3.1), (3.2), (3.3). The equality (3.4) is immediate by inspection. We start with an observation about rules (iiib) and (ivb).

Lemma 8. *If rule (iiib) or (ivb) has been applied to the triangle $X^{(k+1)}$ then in the triangle $X^{(k)}$ there exists a pair $u_i = v_i$.*

Proof. This is immediate for rule (iiib), since adding 1 to both u'_i and v'_i does not destroy the equality $u'_i = v'_i$.

For rule (ivb) we notice that $n - k = v'_k < u'_k$, and $v'_{k-l} + l < u'_k \leq u'_{k-l}$ imply that $v'_{k-j} < u'_{k-j}$ for $j = 1 \dots l$, therefore the inequality $u'_i = v'_i$ must be realized for some $i < k - l$, and then $u_i = u'_i = v'_i = v_i$ by rule (ivb). \square

3.4.2. *Proof of the algorithm.* We now check successively all rules of the inverse algorithm.

(i) It is clear that the triangle $X^{(k)}$ is Gelfand-Tsetlin.

We have $u'_0 = u'_0 - u'_k + v'_k \leq n - 1$, this proves (3.1).

Since $u_0 - 1 - u_i + v_i - (n - k) = u'_0 - u'_i + v'_i - v'_k \leq k - 2$ we have $u_0 - u_i + v_i \leq n - 1$.

All other inequalities in (3.2), (3.3) involve differences like $u_0 - u_i$ or $v_i - v_j$ which are not unchanged by the replacement $u' \rightarrow u, v' \rightarrow v$.

Also inequalities (3.8) are immediate.

(ii) Since $v'_i < u'_i$ for all i one has $v_i \leq u_i$, hence $X^{(k)}$ is a Gelfand-Tsetlin triangle, and (3.1) is immediate since $u_0 = u'_0$.

Since $u'_0 - u'_i + v'_i - v'_k \leq k - 2$ one has $u_0 - u_i + v_i \leq n - 1$, thus (3.2), holds.

Finally (3.3) comes from $u'_0 - u'_i = u_0 - u_i$ $v'_i - v'_j = v_i - v_j$.

The inequalities (3.8) at the next step are immediate.

(iiia) Again it is easy to see that $X^{(k)}$ is a Gelfand-Tsetlin triangle. Since $u'_0 - u'_k + v'_k \leq n - 1$ and $u'_k = v'_k$ we get $u_0 = u'_0 + 1 \leq n$, hence (3.1).

The other inequalities (3.2), (3.3) are checked similarly.

The inequalities (3.8) at the next step are immediate.

(iiib) The fact that $X^{(k)}$ is a Gelfand-Tsetlin triangle is immediate.

Since there exists j with $u'_j = v'_j$ one has $u'_0 = u'_0 - u'_j + v'_j \leq n - 1$ thus $u_0 = u'_0 + 1 \leq n$.

Since $u'_0 - u'_i + v'_i - v'_k \leq k - 2$, it follows that $u'_0 - u'_i + v'_i \leq n - 2$ and $u_0 - u_i + v_i \leq n - 1$.

The other inequalities are satisfied since $u'_0 - u'_i + v'_i - v'_j = u_0 - u_i + v_i - v_j$ for $1 \leq i < j \leq k - 1$.

We now check the inequalities (3.8).

One has $b_k = a_k < a_{k-1} = u'_{k-1} + 1$.

It remains to see that $b_k \leq b_{k-1}$.

If $v'_{k-1} > n - k$, then $v_{k-1} = v'_{k-1} + 1 \geq u'_k = a_k = b_k$ since we are applying rule (iiib) to $X^{(k+1)}$ (in this case, $l = 1$). At the next step, we will have $b_{k-1} \geq v_{k-1} \geq b_k$.

If $v'_{k-1} = n - k$, then one has $l > 1$, and by Lemma 8 either rule (iiib) or rule (ivb) applies to $X^{(k)}$. In either case it is easy to see that $b_k \leq b_{k-1}$.

(iva) In this case, the fact that $X^{(k)}$ is a Gelfand-Tsetlin, as well as the inequalities (3.1), (3.2), (3.3), is immediate. Also the inequalities (3.8) are immediate.

(ivb) Since $n - k < u'_k \leq u'_i$ for $i \leq k - 1$ it follows that $u_i \geq n - k + 1$ for all i . It is then clear that $X^{(k)}$ is a Gelfand-Tsetlin triangle.

Let us check the inequalities (3.1), (3.2), (3.3) for $X^{(k)}$.

Since $u_0 = u'_0$ inequality (3.1) is obvious.

One has $u_0 - u_i + v_i = u'_0 - u'_i + v'_i \leq n - 1$ for $i < k - l$. For $k > i \geq k - l$ one has $v_i = n - k + 1 \leq v_{k-l} + l < u_k \leq u_i$ therefore $-u_i + v_i \leq -1$ and inequality (3.2) holds.

Inequality $u_0 - u_i + v_i - v_j + 1 = u'_0 - u'_i + v'_i - v'_j + 1 \leq j - 1$ holds if $i < j < k - l$.

If $i < k - l$, one has

$$u_0 - u_i + v_i - (n - k) + 1 = u'_0 - u'_i + v'_i - v'_{k-l+1} + 1 \leq k - l$$

hence

$$u_0 - u_i + v_i - v_{k-l} + 1 = u_0 - u_i + v_i - (n - k + 1) + 1 \leq k - l - 1$$

which proves (3.3) for $i < j = k - l$.

If $i < k - l < j$ then $v_i = v'_i$ and $v_j \geq v'_j$ therefore the (3.3) also holds.

One has

$$u_0 - u_{k-l} + v_{k-l} - v_j + 1 \leq u'_0 - u'_{k-l} + v'_{k-l} - v_j + 1 \leq k - l - 1$$

proving (3.3) for $i = k - l < j$.

If $k - l < i < j$, then $v_i = v_j$, and $v'_i = v'_j$ therefore

$$u_0 - u_i + v_i - v_j + 1 = u'_0 - u'_i + v'_i - v'_j + 1 \leq j - 1.$$

It remain to check inequalities (3.8).

After rule (ivb) is applied one has $v_{k-1} = n - k + 1$ and, for some $i < k - 1$, $u_i = v_i$ therefore rule (iiib) or (ivb) applies to the next step. In either case one has $b_k < a_{k-1}$.

Recall that

$$b_k = v'_{k-l} + l < u'_k = a_k$$

and

$$v_{k-1} = \dots v_{k-l} = n - k + 1.$$

It follows that $l' = 1 + \max\{i | v_{k-1-i} = n - k + 1\} \geq l$.

If $v_{k-1-l'} + l' < u_{k-1}$ then rule (ivb) applies to $X^{(k-1)}$ and

$$b_{k-1} = v_{k-1-l'} + l' \geq v'_{k-l} + l = b_k.$$

If $v_{k-1-l'} + l' \geq u_{k-1}$ then $l = l'$, $u_k = u_{k-1}$ and

$$b_{k-1} = v_{k-1-l'} + l' = v_{k-1-l} + l = u_{k-1} > b_k.$$

□

3.5. Some properties of the bijection.

3.5.1. *(n, 1) trapezoids.* If one starts from a $(n, 1)$ trapezoid, then only rules (i) and (ii) apply, and it is easy to see that one gets in the end a $(n, 1)$ GOGAm trapezoid, and that it is obtained by subtracting from any entry of the Gog trapezoid the number of inversions which cover it. The same remark applies to the inverse map, so that our bijection restricts to a bijection between $(n, 1)$ trapezoids.

3.5.2. *(n, 2, k) trapezoids.* A $(n, 2, k)$ Gog trapezoid is a $(n, 2)$ Gog trapezoid such that $X_{jj} = n$ for $j \geq k$, and $X_{j,j-1} = \max(X_{k,k-1}, j - 1)$ for $j \geq k$.

A $(n, 2, k)$ Magog trapezoid is a $(n, 2)$ Magog trapezoid such that $X_{ij} = 1$ for $j \leq k$.

Our bijection restricts to a bijection between $(n, 2, k)$ Gog trapezoids and $(n, 2, k)$ Magog trapezoids for all $k \leq n$.

3.5.3. *A statistic.* For a Gog triangle X the entry X_{11} gives the position of the 1 in the bottom row of the associated alternating sign matrix. If X is a $(n, 2)$ Gog triangle, it follows from our algorithm that the 11 entry of the GOGAm triangle has the same value X_{11} . From Lemma 1 we conclude that for the $(n, 2)$ Magog triangle T , associated to X , one has $X_{11} = \sum_{i=1}^n T_{i,n} - \sum_{i=1}^{n-1} T_{i,n-1}$. it is known that, more generally, these two statistics on Gog and Magog triangles coincide (see e.g. [3]).

References

- [1] D. M. Bressoud, *Proofs and Confirmations, The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.
- [2] H. Cheballah, *PhD Thesis*, in preparation.
- [3] T. FONSECA, P. ZINN-JUSTIN, On the Doubly Refined Enumeration of Alternating Sign Matrices and Totally Symmetric Self-Complementary Plane Partitions, *Electron. J. Combin.* **15**:Research Paper 81, 35pp, 2008.
- [4] A. N. KIRILLOV, A. D. BERENSTEIN, Groups generated by involutions, Gelfand-Tsetlin patterns and combinatorics of young tableaux, *St. Petersburg Math. J.* **7**(1) (1996) **77-127**.
- [5] C. Krattenthaler, A Gog-Magog conjecture,
<http://www.mat.univie.ac.at/~kratt/artikel/magog.html>
- [6] Greg Kuperberg, Another proof of the alternating-sign matrix conjecture. *Internat. Math. Notices* (1996), no. 3, 139-150.
- [7] W.H. Mills, D.P. Robbins, and H. Rumsey, Self complementary totally symmetric plane partitions, *J. Combin. Theory Ser. A* **42**(1986), 277-292.
- [8] W.H. Mills, D.P. Robbins, H. Rumsey, Alternating sign matrices and descending plane partitions. *J. Comb. Theory Ser. A* **34** (1983), 340-359.
- [9] D. Zeilberger, Proof of the alternating sign matrix conjecture. *Electronic J. Combin* **3** (1996), R13.

LABORATOIRE D'INFORMATIQUE DE PARIS NORD, UMR 7030 CNRS, UNIVERSITÉ PARIS
13, F-93430 VILLETANEUSE
E-mail address: Hayat.Cheballah@lipn.univ-paris13.fr

CNRS, IGM- UNIVERSITÉ PARIS-EST, 77454 MARNE-LA-VALLÉE CEDEX2, FRANCE
E-mail address: biane@univ-mlv.fr